The underestimated virtues of the two-sector AK model

by

Gabriel J. Felbermayr*)
Omar Licandro

Working Paper 0315
December 2003

Johannes Kepler University of Linz
Department of Economics
Altenberger Strasse 69
A-4040 Linz - Auhof, Austria
www.econ.jku.at

*)corresponding author:
gabriel.felbermayr@jku.at
phone +43 (0)70 2468 -8265, -8238 (fax)

The underestimated virtues of the two-sector AK model

Gabriel J. Felbermayr[(a)] Omar Licandro[(b)*]

December 2003

Abstract

We show that the two-sector version of the AK model proposed by Rebelo (1991) can be read as an endogenous growth extension of Greenwood, Hercowitz and Krusell (1997). By confining constant returns to capital to the investment goods sector, the model generates endogenously the secular downward trend of the relative price of equipment investment and the rising real investment rate observed in US NIPA data. Whereas Jones (1995) criticizes that the one-sector model fails to reconcile the empirical facts of trending real investment rates and stationary output growth, this incompatibility vanishes in the two-sector version. Finally, a simple technological shock can reproduce the '1974' break in post World War II US data. Thus, AK-type endogenous growth models comply much better with empirical evidence, once they are augmented with a strictly concave consumption sector.

Keywords: AK model; embodiment; endogenous growth; obsolescence; '1974'.
JEL - codes: O41, O30.

*(a) European University Institute and JKU Linz, (b) European University Institute and FEDEA.
We would like to thank Philippe Aghion, Jess Benhabib, Giuseppe Bertola, Antonio Ciccone, Larry Jones, Tim Kehoe, Willi Kohler, Victor Ríos-Rull, Jaume Ventura and Gianluca Violante for helpful comments and suggestions. Moreover, the authors benefitted from discussions at the 7th Macro Workshop at the University of Vigo, the SET Summerschool 2002 in Venice, the 2002 meeting of LACEA in Madrid and the European Conference on Dynamic Macroeconomics at the EUI and presentations at Pompeu Fabra, Tor Vergata, the universities of Louvain and Linz.
Corresponding author: Gabriel Felbermayr, Johannes Kepler University Linz, Economics Department, Altenbergerstraße 68, 4040 Linz, Austria. Phone 0043 732 2468 8265, e-Mail: gabriel.felbermayr@jku.at.
Omar Licandro acknowledges the financial support of the Spanish Ministery of Sciences and Technology (SEC 2000-0260).

1 Introduction

The standard work horses of modern growth theory typically builds on the famous *Kaldor* [10] stylized facts, which postulate, among other things, the stationarity of aggregate ratios such as the investment to output ratio or the capital to output ratio. However, there is now a rich body of empirical research suggesting that some of these alleged facts are in conflict with U.S. evidence. Based on data from the National Income Product Accounts (NIPA), Whelan [22] provides evidence for the post world war II period[1]. His findings can be summarized as follows:

(i) The price of equipment investment relative to the price of consumer nondurables and services has been declining permanently over the post world war II period.

(ii) Nominal series of consumption and investment share a common stochastic trend with nominal output so that the consumption and investment shares in nominal output are stationary.

(iii) The ratio of real equipment investment to real output is non-stationary. Indeed, the growth rate of real equipment investment has been larger than that of real non-durable consumption, reconciling facts (i), (ii) and (iii).[2]

Recent research addresses the inconsistency of facts (i) and (iii) with standard models of exogenous or endogenous growth, in which relative prices are not allowed to change along the balanced growth path. Clearly, to do justice to the new evidence the appropriate framework needs more than just one sector and has to carefully model the sectoral incidence of technological change. Greenwood et al. [5], henceforth abbreviated GHK, based on the seminal contribution of Solow [21], were the first to make an attempt

[1] See also Greenwood, Hercowitz and Krusell [5], p. 342. In many OECD countries, similar trends are visible (see OECD [18]).

[2] Jones [9] uses this observation as a point of departure for his influential critique of AK-type endogenous growth models.

in this direction. Their model consists of a consumption goods sector, which benefits from exogenous disembodied technical progress, and of an investment goods sector, the efficiency of which also grows at an exogenous rate. Sectoral production functions take capital and labor as arguments and are identical up to a Hicks-neutral factor. Advances in the investment sector affect the consumption goods sector to the extent that firms acquire new and more efficient capital goods: technical change is embodied. In this way, GHK are able to generate a permanent decline in the relative price of investment and a rising ratio of real equipment investment to real output, keeping nominal shares constant, as required by the stylized facts cited above. However, in contrast to our model, GHK take growth to be exogenous and cannot analyze the implications of embodied technical change on the incentives to accumulate capital.

We study a version of the two-sector AK model originally proposed by Rebelo [19] (section II), where the consumption goods sector features decreasing returns to capital whilst the investment goods sector is described by an AK technology. We argue that this environment yields the simplest possible endogenous growth model compatible with the evidence cited above. It complements recent research on endogenous embodied technical change in R&D based models of endogenous growth, see Boucekkine et al. [1], Hsieh [8] or Krusell [11], and the model by Boucekkine et al. [2] where technical change stems from learning-by-doing.

In our two-sector model endogenous growth model, where labor and capital are the only factors of production, a decision has to be made concerning the role of capital in the sectoral production functions. The robust and well-known negative correlation between per capita GDP and the relative price of investment goods, observed in cross-sectional studies (see Restuccia and Urrutia [20] for a recent survey) suggests, that – if countries have similar preferences and per capita GDP is closely related to the per capita capital stock – the capital sector has to be the more capital intensive one. Moreover, there is evidence, both formal and informal, that positive production externalities are quantitatively more important in the investment sector than in the consumption sector. Harrison [7] provides econometric evidence on plant-level and sectoral returns to scale in the consumption and investment sectors. She cannot reject constant returns to scale at the *plant*-level in both the consumption and the investment sector, but finds robust evidence for positive externalities at the investment *sector*-level. This coincides with the more informal view that spillovers of many sorts are particularly important in investment goods industries such as the information and communication technologies industry, the car and aeronautic industries, or the industrial equipment industry (see the surveys by OECD [16] and OECD [17]). Thus, it is likely that *social* returns to capital are larger in the investment sector than in the consumption sector. Taking these observations to their extreme, the present model assumes constant marginal returns to capital in the investment sector.

In accordance with GHK's findings, in the proposed framework the relative price of investment trends downward and real investment growth outpaces consumption growth. However, the underlying mechanism is different. GHK generate the price trend by exogenous sectoral differences in the rate of technical progress, whereas our model relates the movement in relative prices to the asymmetric sectoral impact of capital accumulation. Moreover, applying official NIPA methodology, we conclude that real output growth lies above the growth rate of nondurables consumption and below that of equipment investment, which is consistent with fact (iii).

In contrast to its one-sector version, in the two-sector AK model, the user cost of capital is augmented by a capital loss term. This term, that we interpret as obsolescence costs, shows up in any model with a trending relative price of investment, as in the models of embodied technical change proposed by Boucekkine et al. [1] and [2], GHK [5], Hsieh [8] or Krusell [11]. In the proposed AK model, the larger the decline rate of the relative price of investment, the larger the obsolescence cost term. This lowers the real interest rate perceived by consumers and depresses consumption growth.

However, whether the lower interest rate encourages or discourages capital accumulation depends on how the income effect associated with a change in the interest rate relates to the substitution effect, that is, whether the elasticity of intertemporal substitution is smaller or greater than unity. Capital losses reduce (increase) the growth rate of real output if the elasticity of intertemporal substitution is larger (smaller) than the saving rate.

Exploiting the isomorphism of the two-sector AK framework with the social planner version of an economy featuring learning-by-doing in both the investment and the consumption sector, we can interpret the two-sector AK model as a model of *embodied technical change*. With this interpretation in mind, the arrival of a new general purpose technology with a higher relative efficiency of learning in the investment sector boosts *obsolescence* costs and generates a simultaneous increase in the decline rate of the relative price of investment and a reduction in the growth rate of aggregate output, matching the shift in US series experienced around the first oil shock, as reported by Greenwood and Yorukoglu [4]. In the two-sector AK model, this exercise is equivalent to a reduction of the output elasticity in the consumption sector.

The two-sector AK model has an interesting empirical implication. Jones [9] criticizes that the one-sector AK model fails to reconcile the empirical facts of trending real investment rates and stationary output growth. In the two-sector version, where the growth rate is an increasing function of the nominal saving rate but not of the real investment rate, this incompatibility vanishes.

The main conclusion of the paper is that adding a second sector with a strictly concave production function greatly improves the empirical relevance of AK-type endogenous growth models in that it captures features of the recent US growth process in a natural way and fends off the Jones critique.

The remainder of the paper is organized as follows. Section 2 sets out the analytical framework and derives the main propositions; section 3 provides a discussion of the results and compares them with existing models; finally, section 4 concludes.

2 The Model

In this section, we analyze a two-sector, closed economy version of the AK model introduced by Rebelo (1991). The labor force is constant, and all quantities are in per capita terms. The capital stock k_t is endogenously determined by explicit investment decisions. In the consumption sector, capital is combined with labor in a constant returns to scale production function. As in the models of Boucekkine et al. (2002, 2003), Hsieh (2001) and Krusell (1998), the only source of endogenous growth lies in the investment sector. We model this in the simplest possible way, assuming that the technology features constant returns and capital is the only factor of production.

For ease of exposition, we break down the general equilibrium analysis into three parts. First, we characterize the behavior of production firms, who rent capital to produce output. Then, we turn to investment firms, who collect savings, accumulate machinery and rent it out to production firms. Finally, the picture is completed by the optimal savings decision of the representative household.

Production firms. The total capital stock k_t is perfectly mobile intersectorally and can be employed either in the investment or the consumption sector. Using superscripts c and i to denote the respective sectors, full employment implies $k_t = k_t^i + k_t^c$. In the investment sector, per capital output is given by an AK technology $i_t = Ak_t^i$. In the consumption sector, capital and labor are combined following a Cobb-Douglas production function so that $c_t = (k_t^c)^\alpha$, with $\alpha \in]0, 1[$ denoting the output elasticity of capital. Under conditions of perfect competition, firms in each sector rent capital up to the point where the value of its marginal product equals the rental rate R_t. Choosing the consumption good as the numeraire and writing q_t for the relative price of investment goods, capital allocation is determined by the condition

$$R_t = q_t A = \alpha \left(k_t^c\right)^{\alpha-1}. \tag{1}$$

Writing g_x for the proportional rate of growth of variable x, the above condition can be written as

$$\frac{\dot{q}_t}{q_t} = -(1-\alpha)\, g_{k^c}. \tag{2}$$

For condition (1) to hold through time, the relative price q_t has to decrease at that rate at which the marginal productivity of capital in the investment sector falls.

Investment firms. A representative investment firm uses savings of the representative household to purchase investment goods which it rents out to production firms. The investment firm takes the interest rate r_t as given and maximizes the present value of profits $\int_0^\infty (R_t k_t - x_t)\, e^{-\int_0^t r_s \mathrm{d}s} \mathrm{d}t$ subject to the law of motion of capital $\dot{k}_t = x_t/q_t - \delta k_t$ with $k_0 > 0$. The variable x_t denotes investment expenditure and $\delta \in \,]0,1[$ is the rate of physical decay. A straightforward application of the maximum principle requires that the relative price of investment goods has to be equal to the shadow value of capital. Moreover, the rental rate of capital obeys

$$R_t = q_t \left(r_t + \delta - \frac{\dot{q}_t}{q_t} \right). \tag{3}$$

This last equation is central for the understanding of the mechanics of the model. In contrast to models where the relative price of investment goods is constant over time, in our model the capital rental is augmented by capital losses (or gains), as evidenced by the term $\dot{q}_t/q_t$.

Household behavior. The representative household solves

$$\max \int_0^\infty \frac{c_t^{1-\sigma}}{1-\sigma} e^{-\rho t} \mathrm{d}t \tag{4}$$

$$\text{s.t. } \dot{a}_t = r_t a_t + w_t - c_t, \tag{5}$$

with $a_0 > 0$ given. As usual, σ (with $\sigma > 0$ and $\sigma \neq 1$) is the inverse of the intertemporal elasticity of substitution, and $\rho > 0$ denotes the subjective discount rate of the infinitely lived representative individual. Application of the maximum principle delivers the familiar Euler equation

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\sigma}\left(r_t - \rho\right) \tag{6}$$

and the transversality condition $\lim_{t\to\infty} a_t \mu_t e^{-\rho t} = 0$. The costate variable μ_t is equal to $c_t^{-\sigma}$ and evolves according to $\dot{\mu}_t/\mu_t = -\left(r_t - \rho\right)$. μ_t denotes the marginal utility of consumption.

Equilibrium. In general equilibrium, financial wealth has to be equal to the value of the firms, i.e. $a_t = q_t k_t$. The evolution of per capita consumption and the amount of capital invested in the consumption sector can be found by using expression (1) and the law of motion of the relative price of investment goods (2) in the the rental rate (3). This delivers an expression of the equilibrium interest rate

$$r_t = A - \delta + \frac{\dot{q}_t}{q_t}. \tag{7}$$

By substitution, we arrive at an expression for the growth rate of consumption

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\sigma}\left(A - \delta + \frac{\dot{q}_t}{q_t} - \rho\right) \tag{8}$$

Since the value of the invested capital stock falls over time, a capital loss term appears in the rental rate of capital. The larger this term, the lower the interest rate and the flatter the desired consumption profile of the household. Using (2) and recognizing that commodity market clearing requires $g_c = \alpha g_{k^c}$, we find from (8) that

$$g_{k^c} = \frac{1}{\omega}\left(A - \delta - \rho\right), \tag{9}$$

$$g_c = \frac{\alpha}{\omega}\left(A - \delta - \rho\right), \tag{10}$$

where $\omega = 1 - \alpha\left(1 - \sigma\right) > 0$. Both, g_{k^c} and g_c are constant from $t = 0$ on. Substituting $a_t = k_t q_t$, and using expression (3) in (7), the transversality condition characterizing optimal household behavior can be rewritten as

$$\lim_{t\to\infty} \lambda_t k_t e^{-\rho t} = 0, \tag{11}$$

where the shadow value of capital λ_t evolves according to $\dot{\lambda}_t/\lambda_t = -(A-\delta-\rho)$.

Notice that from equation (9), $1/\omega$ is the intertemporal elasticity of substitution associated to k^c. Given our assumptions on technology, we refer to $1/\omega$ as to the *elasticity of intertemporal substitution in foregone investment.* It measures the degree of substitution between using capital to produce consumption goods today and producing investment goods today to produce consumption goods in the future.

Equations (9), (10) and (11) are the key equations of our model.

Assumption 1. *Let the following parameter restriction hold:*

$$A-\delta > \rho > \alpha(1-\sigma)(A-\delta).$$

The first inequality in Assumption 1 is required for the growth rate of consumption, as given by equation (10), to be strictly positive. The second inequality ensures that the utility representation in equation (4) remains bounded in equilibrium.

Proposition 1. *Under Assumption 1, for all $t \geq 0$, the growth rate of capital g_k is constant from $t=0$ onwards so that $k_t = k_0\, e^{g_k t}$. Moreover, a closed form for the consumption function exists and can be written as*

$$c_t = \left(1 - \frac{\delta + g_k}{A}\right)^{\alpha} k_t^{\alpha}.$$

Proof. See the appendix.■

As in the standard AK model, the economy is on its balanced growth path from $t=0$; i.e. there are no transitional dynamics. This is intuitive, since equation (2) requires from $t=0$ on that the relative price of capital adjusts over time so that the value of the marginal product of capital remains constant in both sectors.[3] The

[3] Notice, however, that transition dynamics reappear once the consumption sector technology is modeled using a general CES production function.

equilibrium path of the model economy features $g_k = g_i$ and $g_c = \alpha g_k$. Since $\alpha \in]0,1[$, consumption grows at a slower pace than investment and capital.

As stated in the introduction, in the U.S. three important secular trends are in sharp contradiction with Kaldor's stylized facts. First, the relative price of investment exhibits a secular downward trend. Second, the share of nominal investment in nominal output is constant, and, third, the ratio of equipment investment to real output is steadily increasing. The standard one-sector AK growth model cannot account for these facts. The following proposition shows that a two-sector model with endogenous AK-type growth in the investment goods sector is consistent with these empirical regularities.

Proposition 2. *In the proposed two-sector growth model, (i) the relative price of investment q_t is decreasing at rate $(1-\alpha)\,g_k$, (ii) the nominal saving rate is constant and (iii) the ratio of investment to output is increasing. The growth rate of output, g, defined by a Divisia quantity index, is constant and lies in the interval $]g_c, g_k[$, its exact position being determined by the saving rate.*

Proof.

(i) This just restates equation (2).

(ii) The share of nominal investment in nominal output is given by the saving rate $s_t \equiv q_t i_t / (c_t + q_t i_t)$. Using the results derived above, the saving rate is constant and reads

$$s = \frac{\alpha\,(\delta + g_k)}{A - (1-\alpha)\,(\delta + g_k)}. \tag{12}$$

(iii) In our context, using the Divisia index amounts to writing the growth rate of real output, g, as the weighted sum of the rates of growth of consumption and investment:

$$g = (1-s)\,g_c + s g_i. \tag{13}$$

By Assumption 1, $s \in \left]0,1\right[$ and hence $g \in \left]g_c, g_k\right[$, which implies a permanently increasing investment to output ratio.$\blacksquare$

In Figure 1 we provide a graphical illustration of the equilibrium in the space (i,c). At every point in time, the sectoral production functions and the stock of per capita capital determine a *strictly concave* production possibility frontier. The expansion path Φ shows all pairs of c_t and i_t compatible with the equilibrium in Proposition 1. Φ is found by writing the policy functions $c_t = \phi_c(k_t)$ and $i_t = \phi_i(k_t)$, which are combined by substituting k_t out. In $(i,c)-$ space, Φ can easily be shown to be strictly concave and increasing in i_t. As the economy accumulates capital, it moves north-east along Φ. Equilibrium is identified at the intersection of the transformation curve and the expansion path and is denoted by E. The relative price of investment goods is found as the slope of the transformation curve at point E. Due to the assumed differences in the sectoral production functions, capital accumulation affects sectors asymmetrically and the transformation curve shifts out in an uneven way. This is nothing else than a dynamic version of the familiar Rybczynski effect. Consequently, on the way from E to E', the relative price of investment falls from q to q' at a rate proportional to the rate at which the economy accumulates capital.

As Whelan [22] points out, consistency with NIPA methods requires that the "*ideal chain index*" proposed by Irving Fisher is used to compute the growth rate of real output. This index is a geometric average of a Paasche and a Laspeyres index and can be accurately approximated by the so-called Divisia index, which weights the growth rate of each component of output by its current share in the corresponding nominal aggregate (see Deaton and Muellbauer ([3]), pp. 174-5 for more details). This procedure amounts to continually updating the prices used to calculate real output and to chain the index forward from an arbitrary base year on, in which nominal magnitudes have been set equal to real magnitudes. Moreover, Licandro et al. [13] provide theoretical support for the use of such a chained index in the framework of a two-sector exogenous growth model with embodied technical change (the GHK model). They compute a true quantity index, use official NIPA data to calibrate it for different parameter values, and show that their results come very close to what is obtained by applying NIPA's methodology.

Since expression (13) plays a major role in our analysis, some additional remarks on the appropriate definition of real output growth may be useful. Typically, a base year quantity index is computed as the average growth rate of real output in the different sectors, where base year prices are used as fixed weights. As Whelan [22] explains, in an economy with different sectoral growth rates, such a fixed-weight methodology results in *unsteady* aggregate growth. The growth rate will tend to increase, converging towards the growth rate of the faster growing sector. The reason for this pattern is the so-called substitution bias which occurs when relative prices are held fixed over time. Those categories of output which exhibit faster growth in quantities typically also experience declining relative prices. Measured in prices of a base year, current output will become more and more expensive, as the fast-growing components are still weighted with high historical prices. Moreover, not only does this fixed-weight definition lead to unsteady growth, but the size of the substitution bias, and thus the computed growth rate, depend on the choice of the base-year: the farther in the past, the larger the error. In our model, since the weights used in the computation of the Divisia index are constant, the problem of unsteady growth is avoided.

3 Discussion

In the introduction, we have argued that adding a concave consumption sector to the linear investment goods sector improves the empirical relevance of the AK growth model. This section substantiates this claim. First, we improve the understanding of the enhanced AK model by comparing it to its one-sector analogue. Then we establish a learning-by-doing interpretation of the two-sector AK model. This allows to view our model as a model of embodied technical change. Next, we show that an adverse shock on our parameter α suffices to reproduce the '1974' phenomenon discussed by Greenwood and Yorukoglu [4]. Finally, we review Jones' [9] critique of AK-type endogenous growth models.

Differences to the one-sector version. Setting $\alpha = 1$, the two-sector AK model collapses to the standard one-sector AK model. Then $\dot{q}_t/q_t$ is clearly zero and there are no capital losses along the balanced growth path. Does this imply that the one-sector AK-type economy exhibits faster GDP per capita growth than its two-sector version?

Denote the growth rate of the standard one-sector AK model by g_{AK}. Then the following proposition can be made:

Proposition 3.

$$\begin{array}{rrcl} (i) & g_{AK} & > & g_c, \\ (ii) & g_{AK} & \gtreqless & g_k \Longleftrightarrow \dfrac{1}{\sigma} \gtreqless 1, \\ (iii) & g_{AK} & \gtreqless & g \Longleftrightarrow \dfrac{1}{\sigma} \gtreqless s. \end{array}$$

Proof. In the one-sector AK model the growth rates of consumption, output and capital are all equal to

$$g_{AK} \equiv \frac{1}{\sigma}\left(A - \rho - \delta\right). \tag{14}$$

Parts (i) and (ii) of the proof involve comparing g_{AK} to the growth rates given by equations (10) and (9), which is obvious. To show part (iii), note that g can be expressed as

$$g = \left[\left(1-s\right)\alpha + s\right]\frac{1}{\omega}\left(A - \delta - \rho\right). \tag{15}$$

Then $g_{AK} \gtreqless g \Longleftrightarrow \sigma^{-1} \gtreqless \omega^{-1}\left[\left(1-s\right)\alpha + s\right] \Longleftrightarrow \sigma^{-1} \gtreqless s$. From (9), (13) and the definition of ω, s is a function of σ so that it is not *a priori* clear whether there are values for σ for which the above inequalities hold. The appendix shows that the equation

$\sigma^{-1} = s(\sigma)$ has a unique interior solution $\sigma^{*} > 1$, so that the above inequalities can go either way.■

Note the role of the intertemporal elasticity of substitution in the comparison between g_{AK} and g_k. In the two-sector model, the return to savings being weighed down by capital losses, the agent chooses g_k larger than g_{AK} if the income effect outweighs the substitution effect, that is, if the intertemporal elasticity of substitution is smaller than unity.

In the comparison of (14) and (15) two differences stand out. The first relates to the terms σ^{-1} and ω^{-1}. These terms measure the willingness to forego consumption today for the sake of increased consumption tomorrow. Clearly, with identical production functions for consumption and investment goods, ω^{-1} and σ^{-1} coincide, which is the case in the one-sector model. The second difference relates to the term $(1-s)\alpha + s$, which shows how the marginal effect of a change in g_k effects g, as implied by our definition of output growth (13). A fraction $1-s$ of capital is allocated to the consumption goods sector where it encounters decreasing returns given by α; the complementary fraction goes to the investment sector where returns are constant. In the one-sector model, this term is equal to unity.

Hence, for $g > g_{AK}$ two conditions must be satisfied: first, the elasticity of intertemporal substitution must lie below unity so that the income effect outweighs the substitution effect, generating a larger growth rate of capital; second, the saving rate has to be large enough so as to give sufficient weight to the investment sector when it comes to determining aggregate output growth.

A model of learning-by-doing. Next, we establish that the two-sector AK model is isomorphic to the social planner version of a model by Boucekkine et al. [2] (hereafter BdL), where endogenous growth is due to learning-by-doing in both the consumption and the investment goods sectors. The technological description of their model is given by

$$
\begin{aligned}
c_t + x_t &= z_t k_t^{\eta}, \qquad (16) \\
i_t &= q_t x_t, \qquad (17)
\end{aligned}
$$

where the efficiency of production increases with cumulated net investment so that $z_t = k_t^{\gamma}$ and $q_t = Ak_t^{\lambda}$. The parameters $\lambda > 0$ and $\gamma > 0$ describe the efficiency of learning in the consumption and in the investment sector, respectively. In order to generate sustained balanced growth, a knife edge condition needs to be imposed: $\lambda + \gamma + \eta = 1$. Thanks to constant returns to scale in the production of consumption goods, equations (16) and (17) can be written as

$$
\begin{aligned}
c_t &= \left(k_t^c\right)^{\gamma+\eta}, \\
i_t &= A\left(k_t - k_t^c\right)^{\lambda+\gamma+\eta} = A\left(k_t - k_t^c\right).
\end{aligned}
$$

After the change $\alpha = \gamma + \eta$, the two-sector AK model perfectly coincides with the optimal growth version of the learning-by-doing model. Thus, the two-sector AK model can be seen as the reduced form of a learning-by-doing model where firms internalize the learning externality.

A model of technical change. GHK [5], p. 349, take the observed downward trend of the price of investment goods relative to the price of consumption goods as evidence for investment specific technical change. In their theoretical model, this price trend is generated by sectoral differences in the exogenous rates of technical progress. Boucekkine et al. [1], Hsieh [8] or Krusell [11] take up this idea and write up models where R&D driven endogenous technical progress is confined to the investment goods sector.

In our model, instead, capital accumulation plays a key role. As the economy becomes ever more capital abundant, the capital intensive investment goods sector must expand overproportionally, leading to a decrease in the relative price of investment goods: along the balanced growth path, the marginal rate of transformation falls at a steady rate, as Figure 1 makes clear. This mechanism can be interpreted in two ways. Taking the AK technology in the investment sector literally, the transformation curve changes over time due to capital accumulation and not to technical progress. However, recognizing that the AK description is a reduced form production function of some more complicated model, as discussed above, technical change occurs through changes in the efficiency parameter q_t.

Note that along the balanced growth path the marginal productivity of capital in the consumption goods sector falls, whereas that of the investment goods sector remains constant at A. Therefore, in stark contrast to the above-mentioned R&D models, in the two-sector AK model the relative price trend is driven not by decreasing marginal costs in the investment goods sector, but by increasing marginal costs in the consumption goods sector. This is why we prefer to talk about technical *change* rather than technical *progress*.

The growth effects of embodied technical change. Once the two-sector model is interpreted as a model of technical change, we can inquire about the growth effects of embodied, i.e. investment-specific, technical change. In the proposed modeling framework, the growth rate of real output and the speed of embodied technical change are not necessarily positively related. The reason is that growth rates depend on the difference between the return to savings and the subject discount rate. As cheaper investment goods keep coming into the market, the value of installed capital depreciates. These obsolescence costs drive up the rental rate of capital, and – in face of a constant marginal productivity of capital – reduce the equilibrium interest rate. While the typical intertemporal substitution effect of a lower interest rate unambiguously leads to a flatter consumption path, the growth rate of the capital stock may fall or rise, depending on whether the intertemporal elasticity of substitution is larger or smaller than unity. As a consequence, an increased rate of embodied technical change reduces the growth rate of consumption, has ambiguous effects on the growth rate of investment and thus on the growth rate of real GDP.

'1974' We can now apply the simple argument of the preceding paragraph to the US growth experience. In the learning-by-doing model, the efficiency of learning in the investment goods sector is inversely related to the elasticity of capital in the consumption sector in the two-sector AK model since $\lambda = 1 - \alpha$. From equations (9) and (2) and after substituting $\alpha = 1 - \lambda$, the decline rate of the relative price of investment can be written as

$$\frac{\dot{q}}{q} = -\frac{A - \delta - \rho}{1 + \sigma\left(\frac{1}{\lambda} - 1\right)},$$

which is an increasing function of the efficiency of learning in the investment goods sector relative to the consumption goods sector. Therefore, an adverse shock on the parameter α in the two-sector AK model is equivalent to a positive shock on λ in BdL's framework, with returns to capital in the investment sector kept constant. In this case, reducing the learning efficiency in the consumption sector comes with increasing it in the investment sector. Thus, we can view an adverse shock on α as the arrival of a new general purpose technology which affects both sectors' production functions by *reassigning* sectoral learning efficiencies. Such a shock can realistically account for the change in the US growth pattern observed around the year 1974 and discussed by Greenwood and Yorukoglu [4]: a deceleration in the growth rates of output and consumption, an increase in the growth rate of investment and an acceleration of the decline of the relative price of investment goods.

Consider an unexpected negative shock on the output elasticity of capital in the production function of the consumption goods sector. From equation (2) it can be seen that the decline rate of the relative price of investment increases if α is reduced, since $\partial \left|\dot{q}_t/q_t\right| / \partial\alpha = -g_k\sigma/\omega < 0$. The growth rate of consumption, g_c, is also reduced since $\partial g_c/\partial\alpha = g_k\omega^{-1} > 0$. The effect on g_k is ambiguous and depends on σ^{-1} being smaller or larger than unity: $\partial g_k/\partial\alpha = g_k(1-\sigma)/\omega$. If the income effect of a reduced interest rate dominates the substitution effect ($\sigma^{-1} < 1$), the drop in α increases g_k. However, from equation (13), in order for the shock to generate a reduction in g, σ^{-1} must not be lower than some threshold $\underline{\sigma}^{-1}$. Thus, if $\sigma \in \left]\underline{\sigma}^{-1}, 1\right[$, a reduction in α reproduces the facts put forward by Greenwood and Yorukoglu and can be interpreted as a reassignment of learning efficiency. Note that our account of '1974' relies on a restriction on the intertemporal elasticity of substitution which is empirically plausible.[4] Clearly, the model is too simplistic to claim any fine-tuned realism. However, the ease with which the two-sector AK model captures the broad picture of the recent US growth experience is rather surprising.

A rebattle of the Jones critique. Using data from 1950 to 1987 for 15 OECD countries, Jones [9] criticizes that the standard AK model cannot account for the observed coincidence of stationary growth rates and upward-trending real investment rates. However, his critique is valid only for the one-sector model. The two-sector version reconciles stationary output growth with trending investment rates and overcomes Jones' criticism. Thus, the inconsistency detected by Jones cannot discredit the use of an AK specification in the core investment sector, but merely questions the use of models which keep the price of investment relative to consumption goods constant over time.[5]

4 Conclusion

We analyze a version of the two-sector AK model proposed by Rebelo [19] (section II), where constant aggregate returns to capital are confined to the investment goods sector.

[4] A large body of econometric evidence finds values for the elasticity of intertemporal substitution consistently below unity (see Hall [6] or Ogaki and Reinhart [15] and the references therein). $\underline{\sigma}^{-1}$ is a complicated function of model parameters. It is easy to construct quantitative examples, where the restriction on σ holds and the US growth pattern is approximately reproduced.

[5] See McGrattan [14] and Li [12] for recent econometric attempts to challenge Jones' critique within the standard AK model.

We show that this setup, an endogenous growth extension to the model of Greenwood et al. [5], fits the following empirical observations. First, it provides an endogenous growth rationale for the secular downward trend of the price of investment relative to consumption and the increasing ratio of real investment to real output observed in US NIPA data. Second, again in line with evidence, real output grows faster than consumption but more slowly than investment. Third, an adverse shock on the output elasticity of capital in the consumption sector can be interpreted as a reassignment of relative sectoral learning efficiencies. Thus, a simple technological shock suffices to reproduce the '1974' phenomenon documented by Greenwood and Yorukoglu [4]. Fourth, since the model is compatible at the same time with stationary output growth and upward-trending real investment, it overcomes Jones' [9] well-known critique of the AK model.

Therefore, despite their extreme simplicity, two-sector AK-type endogenous growth models comply much better with empirical evidence once they are augmented with a strictly concave consumption sector. Due to this surprising success, our paper is a contribution towards a defense of AK-type models of endogenous growth.

References

[1] Boucekkine, Raouf, Fernando del Río and Omar Licandro, *Obsolescence and Modernization in the Growth Process*, EUI-ECO Working Papers **2001/18** (2001).

[2] ______, *Embodied Technological Change, Learning-by-doing and the Productivity Slowdown*, Scandinavian Journal of Economics **105** (2003), no. 1, 87–98.

[3] Deaton, Angus and John Muellbauer, *Economics and Consumer Behavior*, Cambridge University Press, Cambridge, UK, 1980.

[4] Greenwood, Jeremy and Mehmet Yorukoglu, *1974*, Carnegie-Rochester Conference Series on Public Policy **46** (1997), 49–95.

[5] Greenwood, Jeremy, Zvi Hercowitz and Per Krusell, *Long-Run Implications of Investment Specific Technological Change*, American Economic Review **87** (1997), no. 3, 342–362.

[6] Hall, Robert E., *Intertemporal Substitution in Consumption*, Journal of Political Economy **96** (1988), no. 2, 339–357.

[7] Harrison, Sharon G., *Returns to Scale and Externalities in the Consumption and Investment Sectors*, Review of Economic Dynamics (forthcoming).

[8] Hsieh, Chang-Tai, *Endogenous Growth and Obsolescence*, Journal of Development Economics **66** (2001), no. 1, 153–171.

[9] Jones, Charles I., *Time Series Tests of Endogenous Growth Models*, Quarterly Journal of Economics **110** (1995), no. 5, 495–525.

[10] Kaldor, Nicholas, *A Model of Economic Growth*, Economic Journal **67** (1957), 591–624.

[11] Krusell, Per, *Investment-Specific R and D and the Decline in the Relative Price of Capital*, Journal of Economic Growth **3** (1998), no. 2, 131–142.

[12] Li, Dajin, *Is the AK Model Still Alive? The Long-run Relation between Growth and Investment Re-examined*, Canadian Journal of Economics **35** (2002), no. 1, 92–114.

[13] Licandro, Omar, Javier Ruiz-Castillo and Jorge Durán, *The Measurement of Growth under Embodied Technical Change*, Recherches Economiques de Louvain **68** (2002), no. 1-2, 7–19.

[14] McGrattan Ellen R., *A Defense of AK Growth Models*, Federal Reserve Bank of Minneapolis Quarterly Review **22** (1998), no. 4, 13–27.

[15] Ogaki, Masao and Carmen M. Reinhart, *Measuring Intertemporal Substitution: The Role of Durable Goods*, Journal of Political Economy **106** (1998), no. 5, 1078–1098.

[16] Organization for Economic Cooperation and Development (OECD), *A New Technology? The Changing Role of Innovation and Information Technology in Growth*, OECD, Paris, 2000a.

[17] ———, *OECD Economic Outlook No. 67*, OECD, Paris, 2000b.

[18] ———, *Saving and Investment: Determinants and Policy Implications*, Economic Outlook (2001 (December)), Section IV.

[19] Rebelo, Sergio, *Long-Run Policy Analysis and Long-Run Growth*, Journal of Political Economy **99** (1991), no. 3, 500–521.

[20] Restuccia, Diego and Carlos Urrutia, *Relative Prices and Investment Rates*, Journal of Monetary Economics **47** (2001), no. 1, 93–121.

[21] Solow, Robert, *Investment and Technological Progress*, Mathematical Methods in the Social Sciences (Stanford, CA) (Kenneth J. Arrow, Samuel Karlin, and Patrick Suppes, ed.), Stanford University Press, 1960, pp. 89–104.

[22] Whelan, Karl, *A Two-Sector Approach to Modeling U.S. NIPA Data*, Journal of Money, Credit, and Banking **35** (2003), no. 4, 627–656.

A Proof of proposition 1

We start from equation (5) and substitute $a_t = q_t k_t$, $w_t = (1-\alpha)\left(k_t^c\right)^{\alpha}$ and $c_t = \left(k_t^c\right)^{\alpha}$. Then, the law of motion of the per capita capital stock turns out to be

$$\dot{k}_t = (A-\delta)\, k_t - A k_t^c.$$

Using $e^{-(A-\delta)t}$ as the integrating factor, substituting $k_t^c = k_0^c e^{g_k t}$ and rearranging terms yields

$$e^{-(A-\delta)t}\left[\dot{k}_t - (A-\delta)\,k_t\right] = -e^{-(A-\delta-g_k)t} A k_0^c.$$

The LHS can easily be recognized as $\frac{d}{dt}\left[e^{-(A-\delta)t}k_t\right]$ and the RHS as $\frac{d}{dt}\left[\frac{1}{(A-\delta-g_k)}e^{-(A-\delta-g_k)t}Ak_0^c\right]$. Integrating and dividing by $e^{-(A-\delta)t}$ gives

$$k_t = \frac{1}{(A-\delta-g_k)} e^{g_k t} A k_0^c + C e^{(A-\delta)t} \tag{A1}$$

where C and k_0^c are constants which can be determined using the initial condition $k_0 > 0$ and the transversality condition 11. Using expression (A1) and the law of motion for the state variable in the transversality condition we get

$$\lim_{t\to\infty}\left\{\frac{A}{(A-\delta-g_k)}e^{-(A-\delta-g_k)t} + C\right\} = 0. \tag{A2}$$

The first term in the braced brackets converges towards zero since $A-\delta-g_k > 0$. Therefore, the TVC requires the constant C to be zero. From (A1) we get $k_0 = Ak_0^c/\left(A-\delta-g_k\right)$.$\blacksquare$

B Proof of proposition 3(iii)

It still needs to be shown that there exist intervals of values for which $\sigma^{-1} \lesseqgtr s$. In particular, it is not clear a priori whether there are values for σ such that $g > g_{AK}$. We need to prove that the equation $\sigma^{-1} = s$ has a unique solution σ^*. Clearly, s is a continuous function. Moreover it is decreasing in σ since

$$\frac{\partial s}{\partial \sigma} = -\left[\frac{\alpha}{A-(1-\alpha)(\delta+g_k)}\right]^2 \frac{A}{\omega} g_k < 0. \tag{A3}$$

Moreover, as σ tends towards infinity, s converges towards the constant

$$s^- = \frac{\alpha\delta}{A-\delta+\alpha\delta} \tag{A4}$$

strictly larger than zero (by Assumption 1) and as σ tends towards zero, s converges towards the constant

$$s^{+} = \frac{\alpha}{1-\alpha} \frac{A - (\rho + \alpha\delta)}{\rho + \alpha\delta} > s^{-}. \tag{A5}$$

Therefore it is clear that the there is a solution σ^{*} to the equation $\sigma^{-1} = s$ and that this solution is bounded below by $(s^{+})^{-1}$. Since $\partial^2 s/\partial\sigma^2 > 0$, we can exclude oscillating behavior of $s(\sigma)$ around σ^{-1}, which is necessary and sufficient for uniqueness of σ^{*}. We conclude that if $\sigma < \sigma^{*} \iff \sigma^{-1} > s \iff g_{AK} > g$ and if $\sigma > \sigma^{*} \iff \sigma^{-1} < s \iff g_{AK} < g$.■

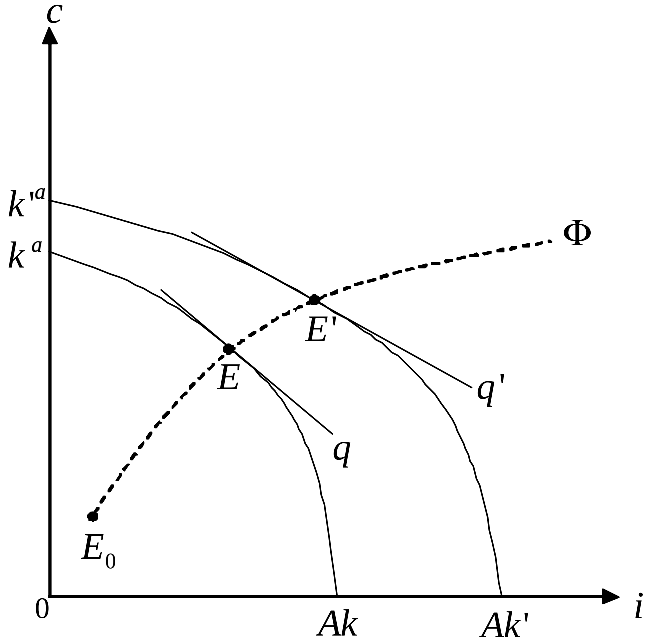

Figure 1: The economy on its expansion path Φ from E to E' $(k' > k)$.